

Periodic Convolution (Circular Convolution)

* Let $x(n)$ and $h(n)$ be two periodic sequences where N is a period for BOTH sequences.

Then

$$y(n) = x(n) \circledast h(n) = \sum_{k=0}^{N-1} x(k) h(n-k) \\ = \sum_{k=0}^{N-1} x(n-k) h(k)$$

is periodic Convolution of $x(n)$ and $h(n)$ with period N . Periodic Convolution is also called Circular Convolution.

Note 1 Fundamental period of $x(n)$ may be N_x and the fundamental period of $h(n)$ may be N_h .

However, the N must be a period for both $x(n)$ and $h(n)$, i.e. smallest integer multiple of N_x and N_h .

Note 2 $y(n)$ is periodic too with period N .

Note 3 Since $x(n)$, $h(n)$, and $y(n)$ are all periodic with period N , then the shift in the periodic (or circular) convolution is within the interval of 0 to $N-1$, since

$$y(lN + \alpha) = y(\alpha) \quad \text{for } 0 \leq \alpha \leq N-1$$

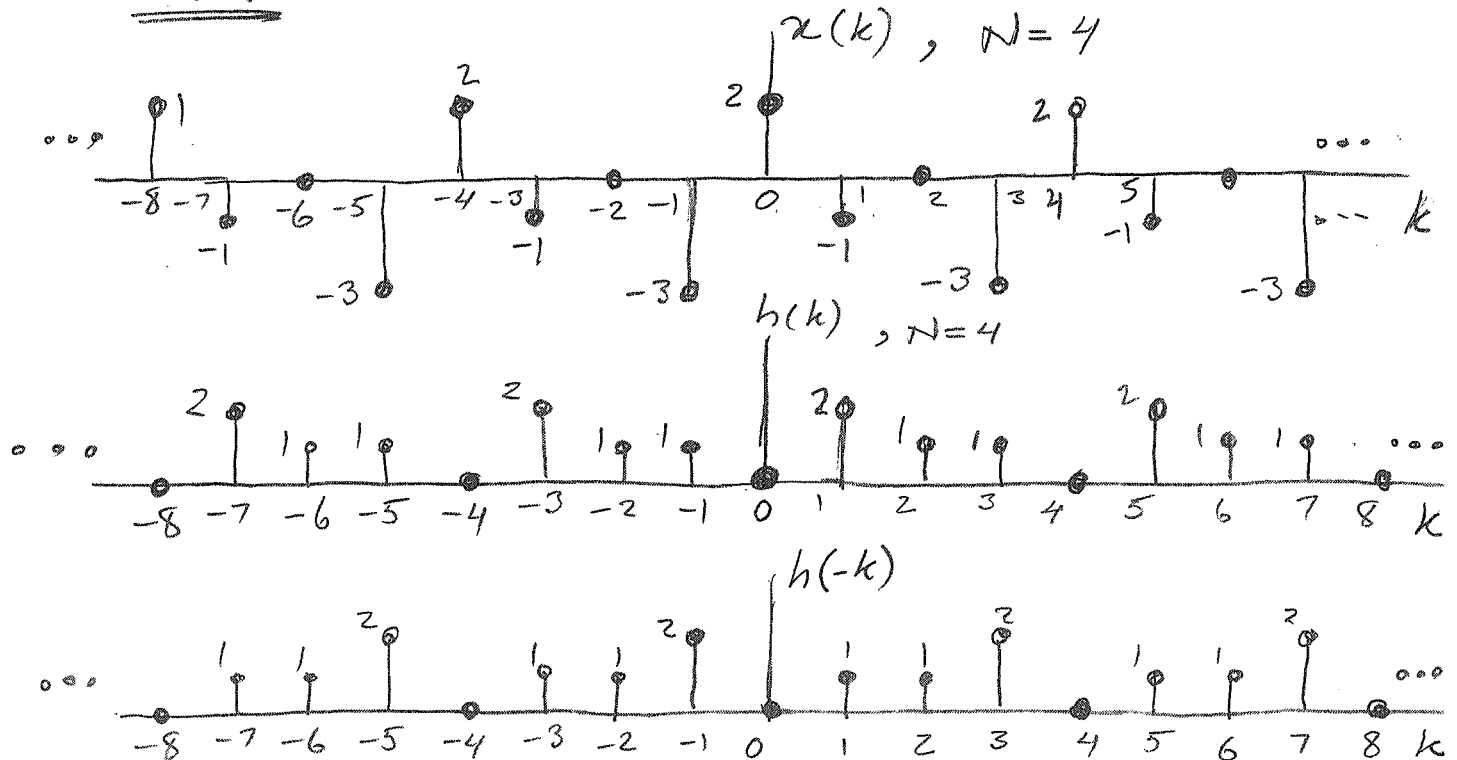
l is any integer.

$$= \sum_{k=0}^{N-1} x(k) h(lN + \alpha - k)$$

$$= \sum_{k=0}^{N-1} x(k) h(\alpha - k)$$

$$= \sum_{k=0}^{N-1} x(\alpha - k) h(k)$$

Ex 1



$$n=0 \quad y(0) = \sum_{k=0}^3 x(k) h(-k) = x(0)h(0) + x(1)h(-1) + x(2)h(-2) + x(3)h(-3)$$

$$y(0) = 2(0) + (-1)(1) + (0)(1) + (-3)(2)$$

$$y(0) = 0 - 1 + 0 - 6 = -7$$

$$n=1 \quad y(1) = \sum_{k=0}^3 x(k)h(1-k) = x(0)h(1) + x(1)h(0) + x(2)h(-1) + x(3)h(-2)$$

$$y(1) = (2)(2) + (-1)(0) + (0)(1) + (-3)(1)$$

$$y(1) = 4 - 3 = +1$$

$$n=2 \quad y(2) = \sum_{k=0}^3 x(k)h(2-k) = x(0)h(2) + x(1)h(1) + x(2)h(0) + x(3)h(-1)$$

$$y(2) = (2)(1) + (-1)(2) + (0)(0) + (-3)(1)$$

$$y(2) = 2 - 2 - 3 = -3$$

$$n=3 \quad y(3) = \sum_{k=0}^3 x(k)h(3-k) = x(0)h(3) + x(1)h(2) + x(2)h(1) + x(3)h(0)$$

$$y(3) = (2)(1) + (-1)(1) + (0)(2) + (-3)(0)$$

$$y(3) = 2 - 1 = 1$$

$$n=4 \quad y(4) = \sum_{k=0}^3 x(k)h(4-k) = \sum_{k=0}^3 x(k)h(-k)$$

$$= x(0)h(4) + x(1)h(3) + x(2)h(2) + x(3)h(1)$$

$$= x(0)h(0) + x(1)h(-1) + x(2)h(-2) + x(3)h(-3)$$

$$y(4) = (2)(0) + (-1)(1) + (0)(1) + (-3)(2)$$

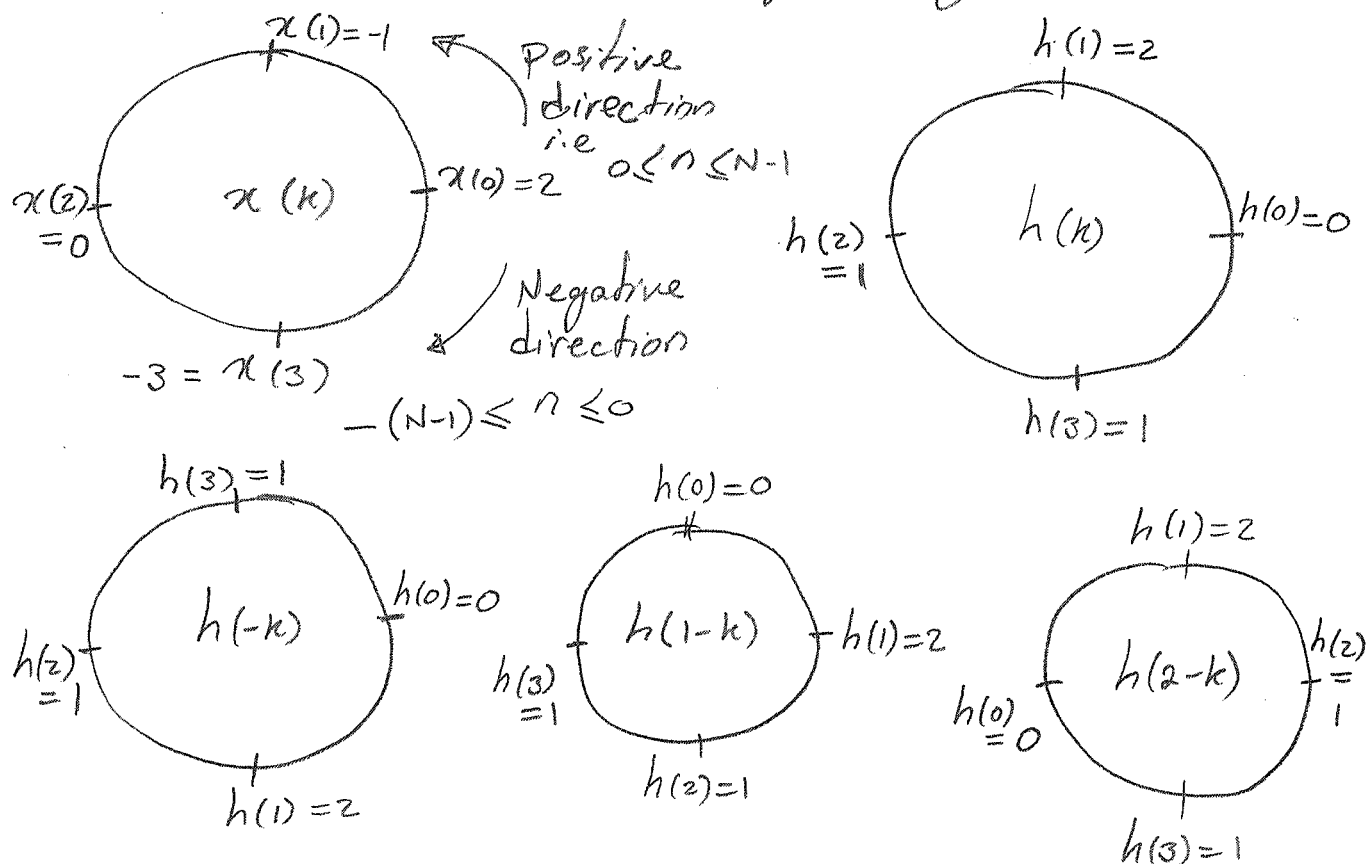
$$= -1 - 6 = -7 = y(0)$$

Since $y(l_N + \alpha) = y(\alpha)$ for $0 \leq \alpha \leq N-1$

* Periodic/circular Convolution can also be done as follows.

Note 1 Given the periodic sequence $h(n) = h(n+N)$

then $h(-n)$ is folded version around $n=0$. This should be noted in the following.



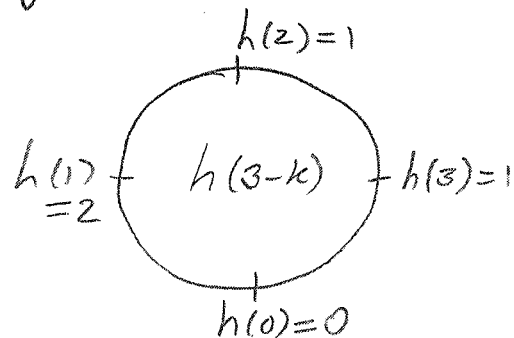
$$y(0) = x(0)h(0) + x(1)h(3) + x(2)h(2) + x(3)h(1)$$

$$y(0) = 0 - 1 + 0 - 6 = -7$$

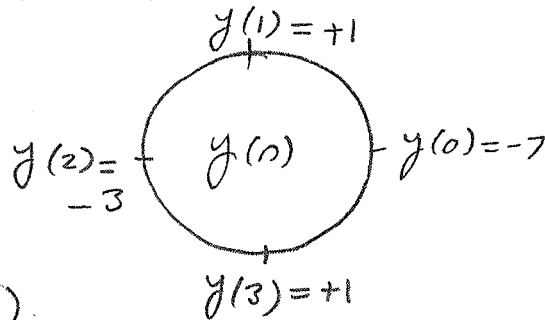
$$y(1) = 4 + 0 + 0 - 3 = +1$$

$$y(3) = 2 - 1 + 0 + 0 = +1$$

$$y(2) = 2 - 2 + 0 - 3 = -3$$



Thus



$$y(n) = y(n+4)$$

periodic with period N .

* Periodic Convolution of 2 finite sequences

Given $x(n)$ of length M , i.e. $0 \leq n \leq M-1$ and

$h(n)$ of length L , i.e. $0 \leq n \leq L-1$,

then periodic or circular Convolution

$$y(n) = x(n) \circledast h(n) \quad (I)$$

Note Can be done for any N .

However if $N \geq L+M-1$ then

the periodic/circular Convolution (I) will be the same as Non-periodic (Non-circular) Convolution.

Note that is we add zeros at the end of the sequences (Zero-padding) $x(n)$ and $h(n)$ making them each of length $N \geq L+M-1$.

Ex $x(n) = \{a, b\}$, $h(n) = \{e, f, g\}$

$\uparrow$ $\uparrow$
 $m=2$ $L=3$

Then for $N \geq (L+m-1=4)$ we can let $N=8$.

Thus

$$x(n) = \{a, b, \underbrace{0, 0, 0, 0, 0, 0}_{\text{Zero-padding}}\}$$

$\uparrow$

$$h(n) = \{e, f, g, \underbrace{0, 0, 0, 0, 0}_{\text{Zero-padding}}\}$$

$\uparrow$

Now

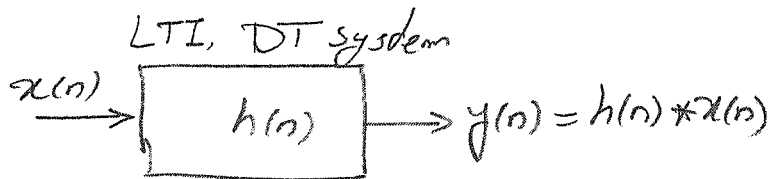
$y_p(n) = x(n) \textcircled{8} h(n)$ is periodic/circular convolution of period N .

The results is the same as Regular, Non-periodic, Linear Convolution

$$y(n) = x(n) * h(n) =$$

$$y(n) = \{ea, \underset{\uparrow}{fa+eb}, ga+fb, eb, 0, 0, 0, 0\}$$

* Causality



The LTI, DT system is Causal if

$$h(n) = 0 \text{ for } n < 0.$$

* Finite Impulse Response (FIR)

The LTI, DT system is FIR if $h(n)$ is a finite length sequence.

* Infinite Impulse Response (IIR)

The LTI, DT system is IIR if $h(n)$ is an infinite length sequence.

Ex1 For Causal FIR system, we have

$$\begin{aligned}
 y(n) &= x(n) * h(n) = \sum_{k=0}^M h(k) x(n-k) \\
 &= \sum_{k=n-M}^n x(k) h(n-k)
 \end{aligned}$$

where

$h(n)$ is of length M and

$h(n) = 0$ for $n < 0$, and $n > M$.

Since for $h(n-k)$, $0 \leq n-k \leq M$.

Ex 2 For Causal IIR system, we have

$$\begin{aligned} y(n) &= x(n) * h(n) = \sum_{k=0}^{\infty} h(k) x(n-k) \\ &= \sum_{k=-\infty}^n x(k) h(n-k) \end{aligned}$$

Since $0 \leq n-k < \infty$.

* BIBO stability

The LTI-DT system is BIBO stable iff

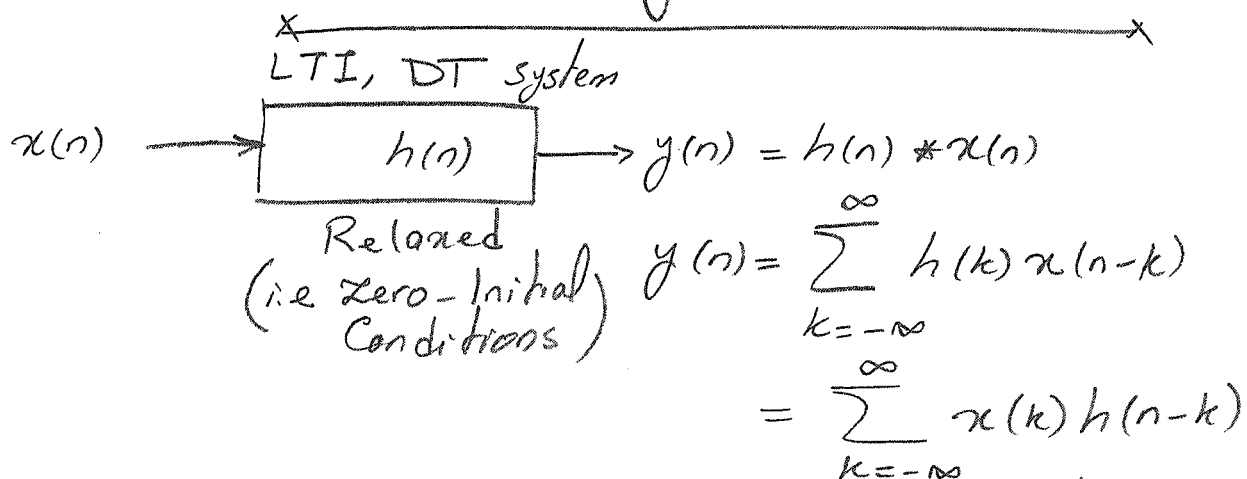
$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty \quad \text{i.e. } h(n) \text{ is absolutely summable.}$$

Note If $h(n)$ is a finite energy signal then it represents a BIBO stable since

$$E_h = \sum_{n=-\infty}^{\infty} |h(n)|^2 < \infty.$$

where $h(n)$ is the Impulse Response of a LTI, DT system.

Difference Equation Representing Input-Output Relation of Discrete-Time Systems



- * The Convolution relation give an Input-output relationship for a LTI, DT system.
- * Another way to represent the Input-output relationship is the Difference Equation.

In general:

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m) \quad (II)$$

a_k and b_m are constant coefficients.

*
$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m) = x(n) * h(n)$$

when Initial Conditions are zero.

$$= \sum_{k=-\infty}^{\infty} h(k) x(n-k)$$

Note 1

Eg. (II) represent a general IIR, Causal, LTI, DT system.

Note 2

If the LTI, DT, Causal system is FIR, then the Difference equation given in (II) becomes

$$y(n) = \sum_{m=0}^M b_m x(n-m) \equiv \sum_{m=0}^M h(m) x(n-m)$$

i.e. $h(n) * x(n)$

Thus: $b_m = h(m)$ for FIR system, for $0 \leq m \leq M$, i.e. $h(n)$ is Causal FIR of length $M+1$.

Note 3

In Eg (II), order of the Difference equation is equal to $\text{Max}\{N, M\}$.

Note 4

At starting time, say at $n=0$, we have from Eg (II):

$$y(0) = - \sum_{k=1}^N a_k y(-k) + \sum_{m=0}^M b_m x(-m)$$

When the input $x(n)$ is applied at $n=0$, then the values of the output $y(-1), y(-2), \dots, y(-N)$

Constitute the initial conditions of the LTI, DT systems.

* Thus the output $y(n)$ is uniquely found

if we have:

(1) The Initial Condition values; $y(-1), y(-2), \dots$
 $\dots, y(-N)$

and

(2) The $a_1, a_2, a_3, \dots, a_N$ and

$b_0, b_1, b_2, \dots, b_M$

and

(3) the input values of $x(n)$.

for any N , and M .

* Iterative way to find $y(n)$ for $n \geq 0$ from the Difference Equation — Often used by Computer algorithms:

Ex1 For the system described by

$$y(n) = a y(n-1) + x(n)$$

we apply input $x(n)$ at time $n=0$ and

want to find the output $y(n)$ for $n \geq 0$.

Note that for this first order difference equation $y(-1)$ is the initial condition value.

$$n=0 \Rightarrow y(0) = a y(-1) + x(0)$$

$$n=1 \Rightarrow y(1) = a y(0) + x(1) = a^2 y(-1) + a x(0) + x(1)$$

$$n=2 \Rightarrow y(2) = a y(1) + x(2) = a^3 y(-1) + a^2 x(0) + a x(1) + x(2)$$

...

$$\text{for } n=n \Rightarrow y(n) = a^{n+1} y(-1) + \sum_{k=0}^n a^k x(n-k)$$

$$y(n) = a^{n+1} y(-1) + \underbrace{[a^n u(n) * x(n) u(n)]}_{\text{Zero-state response}}$$

(IV)

Zero-Input
response

(i.e. for no
input, $x(n)=0$)

Zero-state
response

(i.e. for Zero Initial
Conditions)

* To find Impulse response of this system we let $x(n) = \delta(n)$ and assume Zero-Initial Condition (i.e we assume $y(-1) = 0$).

Then the output $y(n) = h(n)$ is the impulse response of the system.

That is

$$y(n) = h(n) = ay(n-1) + \delta(n) \quad , \quad y(-1) = 0$$

then $h(n) = ah(n-1) + \delta(n)$

$n=0 \quad h(0) = ah(-1) + 1 = 1 \quad \text{for } h(-1) = 0$

$n=1 \quad h(1) = ah(0) = a$

$n=2 \quad h(2) = ah(1) = a^2$

$\vdots$

$n=n \quad h(n) = a^n \quad \text{for } n \geq 0, \text{ The Impulse Response}$

Note

For This system to be BIBO stable we must check the absolute summability of $h(n) = a^n u(n)$

That is

$$\sum_{n=0}^{\infty} |a^n| = \sum_{n=0}^{\infty} |a|^n = \frac{1 - |a|^{\infty}}{1 - |a|} \Big|_{|a| < 1} = \frac{1}{1 - |a|}$$

Thus if $|a| < 1$ then $h(n) = a^n u(n)$ is BIBO stable.

Recalling Eq. (IV):

$$y(n) = \underbrace{a^{n+1} y(-1)}_{\text{Zero-Input response}} + \underbrace{h(n) * x(n)}_{\text{Zero-state response}}$$

← convolution of $x(n)$ and $h(n)$.

Since $|a| < 1$ (for stability) then
as $n \rightarrow \infty$, the Zero-Input response dies out.

Note If the system was relaxed (or at rest, or with zero-initial conditions) then
 $y(n) = h(n) * x(n)$, as mentioned before.

Solving the Difference Equation

That is we want to analytically find a closed form of $y(n)$ as solution to

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m) \quad \text{for } n \geq 0$$

When we have:

and (1) The initial values $y(-1), y(-2), \dots, y(-N)$

(2) a_k 's and b_m 's

and

(3) $x(n)$ for $n \geq 0$.

Sol

Since the system is LTI, we find

(1) Homogeneous solution or response, $y_H(n)$.

and That is the system response when $x(n)=0$

(2) The particular solution or response, $y_p(n)$, with the given input.

and

(3) We use the initial conditions to find the remaining constants.

and

(4) the final response is $y(n) = y_H(n) + y_p(n)$

Step 1: Find $y_H(n)$, $x(n)=0$

then the Homogeneous solution/response is the one satisfying the following Homogeneous equations:

$$y(n) + \sum_{k=1}^N a_k y(n-k) = 0 \quad , \text{ i.e for } x(n)=0 \quad (A)$$

We let $y(n) = \lambda^n u(n)$, so that

$$\lambda^n + \sum_{k=1}^N a_k \lambda^{n-k} = 0 \Rightarrow \lambda^n + a_1 \lambda^{n-1} + a_2 \lambda^{n-2} + \dots + a_N \lambda^{n-N} = 0$$

$$\text{or } \lambda^n \lambda^{-N} \{ \lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \dots + a_N \} = 0$$

Thus we must have:

$$\lambda^N + a_1 \lambda^{N-1} + a_2 \lambda^{N-2} + \dots + a_N = 0 \quad (B)$$

This is "Characteristic Equation". This is a polynomial equation of degree N with N roots $\lambda_1, \lambda_2, \lambda_3, \dots, \lambda_N$ where for

$\lambda = \lambda_k$, $k=1, 2, \dots, N$, the above characteristic equation will be zero.

Thus λ_k^n , for $k=1, 2, \dots, N$, is a solution to the Homogeneous equation (A). Since the system is linear then

$$y_H(n) = c_1 \lambda_1^n + c_2 \lambda_2^n + \dots + c_k \lambda_k^n + \dots + c_N \lambda_N^n$$

is the Homogeneous solution.

Note 1 the unknown parameters C_k , $k=1, 2, \dots, N$, will be defined later using Initial conditions.

Note 2 In Eq. (B) for any repeated root λ_i of multiplicity p , then we have

$$y_H(n) = C_1 \lambda_1^n + C_2 \lambda_2^n + \dots + C_i \lambda_i^n + C_{i+1} n \lambda_i^n + C_{i+2} n^2 \lambda_i^n + \dots + C_{i+p-1} n^{p-1} \lambda_i^n + \dots + C_N \lambda_N^n$$

Step 2 Find $y_p(n)$ using the given input $x(n)$.
To do this, we choose $y_p(n)$ of the same form as $x(n)$.

The Particular solution must satisfy the equation

$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m) \quad (V)$$

when $x(n)$ is in the equation.

* Consider the following table for choosing proper particular solution:

The Input $x(n)$	Take $y_p(n)$ as
$A u(n)$	$k u(n)$
$A \delta(n)$	$k \delta(n)$
$A \beta^n u(n)$	$k \beta^n u(n)$
$A n^M u(n)$	$k_0 n^M + k_1 n^{M-1} + k_2 n^{M-2} + \dots + k_M$
$A e^{j\omega_0 n} u(n)$	$k e^{j\omega_0 n} u(n)$
$A \beta^n n^M u(n)$	$\beta^n [k_0 n^M + k_1 n^{M-1} + k_2 n^{M-2} + \dots + k_M]$
$\left\{ \begin{array}{l} A \cos(\omega_0 n) \\ A \sin(\omega_0 n) \end{array} \right\}$	$K_1 \cos(\omega_0 n) + K_2 \sin(\omega_0 n)$

Note 1

* A is a Constant

* ω_0 is known frequency

* $k, k_0, k_1, \dots, k_M$ are unknown constants to be defined, as follows.

Note 2

If $x(n)$ is any combination of those listed in the above table, then $y_p(n)$ must be taken as the combination of respective particular solutions listed in the table.

Step 3 we now have $y_{\text{tot}}(n) = y_H(n) + y_p(n)$.
Thus

$$y_{\text{tot}}(n) = y(n) \quad \text{for every } n \geq 0.$$

where

$$\begin{cases} y_{\text{tot}}(n) = \sum_{k=1}^N c_k \lambda_k^n u(n) + y_p(n) \\ y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m) \end{cases}$$

↓
given $x(n)$ is
inserted here.

Then, we equate

$$y_{\text{tot}}(n) = y(n) \quad \text{for } n = 0, 1, 2, \dots, N-1$$

to obtain N equations in N unknowns $c_1, c_2, \dots, c_N$.

Solving these, we find $c_1, c_2, \dots, c_N$.

Then

$y_{\text{tot}}(n)$ is the solution to the Difference equation.

Note in defining $y_p(n)$ in Step 2, no term in $y_p(n)$ must be the same as any term in $y_H(n)$. If it is we then modify that term in $y_p(n)$ by putting an n^l in front of it.

$l=p+1$ is greater than the power of n factor, if any, in the corresponding term in $y_H(n)$.

Ex1 Find $y(n)$ for $n \geq 0$ for

$$y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + x(n) - 2x(n-1)$$

where $x(n) = \left(\frac{1}{5}\right)^n u(n)$, $y(-1) = 1$, $y(-2) = -2$.

Sol

Step 1

$$y_H(n), x(n) = 0:$$

$$y(n) - \frac{5}{6} y(n-1) + \frac{1}{6} y(n-2) = 0$$

$$\lambda^2 - \frac{5}{6} \lambda + \frac{1}{6} = 0 \quad \text{characteristic Eq.}$$

$$\left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{3}\right) = 0 \Rightarrow \boxed{\lambda_1 = \frac{1}{2}}, \boxed{\lambda_2 = \frac{1}{3}}$$

Thus

$$y_H(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n \quad \text{for } n \geq 0.$$

Step 2

$$y_p(n), x(n) = \left(\frac{1}{5}\right)^n u(n).$$

Thus we let $y_p(n) = K \left(\frac{1}{5}\right)^n u(n)$.
and inserting in the difference equation, we get

$$K \left(\frac{1}{5}\right)^n u(n) - \frac{5}{6} K \left(\frac{1}{5}\right)^{n-1} u(n-1) + \frac{1}{6} K \left(\frac{1}{5}\right)^{n-2} u(n-2) = \left(\frac{1}{5}\right)^n u(n) - 2 \left(\frac{1}{5}\right)^{n-1} u(n-1)$$

For $n=2$, we get:

$$K \left(\frac{1}{25}\right) - \frac{5}{6} K \left(\frac{1}{5}\right) + \frac{1}{6} K = \left(\frac{1}{25}\right) - 2 \left(\frac{1}{5}\right)$$

$$\text{and } K \left\{ \frac{1}{25} - \frac{1}{6} + \frac{1}{6} \right\} = \frac{-9}{25} \Rightarrow \boxed{K = -9}$$

$$y_p(n) = -9 \left(\frac{1}{5}\right)^n u(n)$$

Steps

$$\begin{cases} y_{\text{tot}}(n) = y_H(n) + y_p(n) = C_1 \left(\frac{1}{2}\right)^n + C_2 \left(\frac{1}{3}\right)^n - 9 \left(\frac{1}{5}\right)^n, \quad n \geq 0 \\ y(n) = \frac{5}{6} y(n-1) - \frac{1}{6} y(n-2) + \left(\frac{1}{5}\right)^n u(n) - 2 \left(\frac{1}{5}\right)^{n-1} u(n-1) \end{cases}$$

$$n=0 \Rightarrow \begin{cases} y_{\text{tot}}(0) = C_1 + C_2 - 9 \\ y(0) = \frac{5}{6} y(-1) - \frac{1}{6} y(-2) + 1 \\ y(0) = \frac{5}{6} + \frac{1}{3} = \frac{7}{6} \end{cases}$$

$$\text{Thus } y_{\text{tot}}(0) = y(0) \Rightarrow C_1 + C_2 - 9 = \frac{7}{6}$$

$$\boxed{C_1 + C_2 = \frac{61}{6}}$$

$$n=1 \Rightarrow \begin{cases} y_{\text{tot}}(1) = \frac{1}{2} C_1 + \frac{1}{3} C_2 - \frac{9}{5} \\ y(1) = \frac{5}{6} y(0) - \frac{1}{6} y(-1) + \frac{1}{5} - 2 \end{cases}$$

$$y(1) = \left(\frac{5}{6}\right)\left(\frac{7}{6}\right) - \frac{1}{6}(1) + \frac{1}{5} - 2 = \frac{-179}{180}$$

Thus

$$\frac{1}{2} C_1 + \frac{1}{3} C_2 - \frac{9}{5} = -\frac{179}{180}$$

$$\boxed{3C_1 + 2C_2 = \frac{54}{5} - \frac{179}{180} = \frac{1765}{180} = \frac{353}{36}}$$

Now

$$\begin{cases} C_1 + C_2 = \frac{61}{6} \\ 3C_1 + 2C_2 = \frac{353}{36} \end{cases} \Rightarrow \begin{cases} C_1 = \frac{-379}{36} \\ C_2 = \frac{745}{36} \end{cases}$$

Thus the final answer is

$$\boxed{y(n) = -\frac{379}{36} \left(\frac{1}{2}\right)^n + \frac{745}{36} \left(\frac{1}{3}\right)^n - 9 \left(\frac{1}{5}\right)^n \text{ for } n \geq 0}$$

Ex 2Find $y(n)$ for $n \geq 0$ for

$$y(n) = -\frac{1}{4}y(n-1) + \frac{1}{8}y(n-2) + x(n)$$

where $x(n) = \left(\frac{1}{4}\right)^n u(n)$, $y(-1) = 0$, $y(-2) = 1$.

SolStep 1

$$y_H(n), \quad x(n) = 0$$

$$y(n) + \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = 0$$

$$\lambda^2 + \frac{1}{4}\lambda - \frac{1}{8} = 0 \Rightarrow \lambda_1 = -\frac{1}{2}, \lambda_2 = \frac{1}{4}$$

Thus

$$y_H(n) = c_1 \left(-\frac{1}{2}\right)^n + c_2 \left(\frac{1}{4}\right)^n \quad \text{for } n \geq 0$$

Step 2

$$y_p(n), \quad x(n) = \left(\frac{1}{4}\right)^n u(n)$$

We notice that the input is of the form of 2nd term of $y_H(n)$. Thus we let

$$y_p(n) = K n \left(\frac{1}{4}\right)^n u(n).$$

To find K :

$$\begin{aligned} K n \left(\frac{1}{4}\right)^n u(n) + \frac{1}{4} K (n-1) \left(\frac{1}{4}\right)^{n-1} u(n-1) - \frac{1}{8} K (n-2) \left(\frac{1}{4}\right)^{n-2} u(n-2) \\ = \left(\frac{1}{4}\right)^n u(n) \end{aligned}$$

for $n=2$, we get

$$k(2)\left(\frac{1}{16}\right) + \frac{1}{4}k(1)\left(\frac{1}{4}\right) + 0 = \left(\frac{1}{4}\right)^2$$

$$\frac{1}{8}k + \frac{1}{16}k = \frac{1}{16} \Rightarrow \boxed{k = \frac{1}{3}}$$

Thus
$$y_p(n) = \frac{1}{3}n \left(\frac{1}{4}\right)^n u(n)$$

Step 3

$$\begin{cases} y_{tot}(n) = c_1 \left(-\frac{1}{2}\right)^n + c_2 \left(\frac{1}{4}\right)^n + \frac{1}{3}n \left(\frac{1}{4}\right)^n \\ y(n) = -\frac{1}{4}y(n-1) + \frac{1}{8}y(n-2) + \left(\frac{1}{4}\right)^n \end{cases}$$

$$\begin{aligned} n=0 \quad & \begin{cases} y_{tot}(0) = c_1 + c_2 \\ y(0) = -\frac{1}{4}y(-1) + \frac{1}{8}y(-2) + 1 \end{cases} \\ & y(0) = \frac{1}{8} + 1 = \frac{9}{8} \end{aligned}$$

Thus
$$\boxed{c_1 + c_2 = \frac{9}{8}}$$

$$\begin{aligned} n=1 \quad & \begin{cases} y_{tot}(1) = -\frac{1}{2}c_1 + \frac{1}{4}c_2 + \frac{1}{4}\left(\frac{1}{3}\right) \\ y(1) = -\frac{1}{4}y(0) + \frac{1}{8}y(-1) + \frac{1}{4} \end{cases} \\ & y(1) = -\frac{1}{4}\left(\frac{9}{8}\right) + \frac{1}{4} = \frac{-1}{32} \end{aligned}$$

$$-\frac{1}{2}c_1 + \frac{1}{4}c_2 + \frac{1}{12} = \frac{-1}{32} \Rightarrow \boxed{-2c_1 + c_2 = \frac{-11}{24}}$$

$$\begin{cases} C_1 + C_2 = \frac{9}{8} \\ -2C_1 + C_2 = \frac{-11}{24} \end{cases} \Rightarrow \boxed{\begin{aligned} C_1 &= \frac{19}{36} \\ C_2 &= \frac{43}{72} \end{aligned}}$$

Thus

$$y(n) = \frac{19}{36} \left(-\frac{1}{2}\right)^n + \frac{43}{72} \left(\frac{1}{4}\right)^n + \frac{1}{3} n \left(\frac{1}{4}\right)^n \quad \text{for } n \geq 0$$

Finding System's Impulse Response from given difference equation

Let
$$y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m)$$

the output is $h(n)$ if

(1) $x(n) = \delta(n)$, and (2) Initial Conditions are zero.

Thus

step 1 $y_H(n), x(n)=0$

$$y_H(n) = \sum_{k=1}^N c_k \lambda_k^n \quad \text{for } n \geq 0$$

step 2 $y_p(n) = K \delta(n)$

$$K \delta(n) + \sum_{k=1}^N a_k K \delta(n-k) = \sum_{m=0}^M b_m \delta(n-m)$$

If $N > M$ then for $n=N \Rightarrow K=0$

$$\Rightarrow y_p(n) = 0.$$

Thus for $N > M \Rightarrow y_H(n) = h(n) = \sum_{k=1}^N c_k \lambda_k^n$
for $n \geq 0$

Step 3

$$\begin{cases} y_{\text{tot}}(n) = h(n) = \sum_{k=1}^N c_k \lambda_k^n \\ y(n) = -\sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m \delta(n-m) \end{cases}$$

where $y(-1) = y(-2) = \dots = y(-N) = 0$

We use the following set of equations to find $c_k, 1 \leq k \leq N$:

$$n=0 \Rightarrow y_{\text{tot}}(0) = y(0)$$

$$n=1 \Rightarrow y_{\text{tot}}(1) = y(1)$$

$\vdots$

$$n=N-1 \Rightarrow y_{\text{tot}}(N-1) = y(N-1)$$

$\star$

Ex 3

Find impulse response of the system in Ex 2 in page 6-53.

Sol

$$y_H(n) = h(n) = c_1 \left(-\frac{1}{2}\right)^n + c_2 \left(\frac{1}{4}\right)^n \quad n \geq 0$$

$$y_p(n) = 0$$

$$\begin{cases} y_{\text{tot}}(n) = h(n) = c_1 \left(-\frac{1}{2}\right)^n + c_2 \left(\frac{1}{4}\right)^n \\ y(n) = -\frac{1}{4} y(n-1) + \frac{1}{8} y(n-2) + \delta(n) \end{cases}$$

$$n=0 \quad \begin{cases} y_{\text{tot}}(0) = c_1 + c_2 \\ y(0) = \frac{1}{8} + 1 = \frac{9}{8} \end{cases} \Rightarrow \boxed{c_1 + c_2 = 9/8}$$

$$n=1 \quad \begin{cases} y_{\text{tot}}(1) = -\frac{1}{2} C_1 + \frac{1}{4} C_2 \\ y(1) = -\frac{1}{4} y(0) = -\frac{1}{4} \left(\frac{9}{8}\right) \Rightarrow \end{cases} \quad \boxed{-2C_1 + C_2 = -\frac{9}{8}}$$

Thus

$$\begin{cases} C_1 + C_2 = \frac{9}{8} \\ -2C_1 + C_2 = -\frac{9}{8} \end{cases} \Rightarrow \boxed{C_1 = \frac{3}{4}, \quad C_2 = \frac{3}{8}}$$

Thus

$$\boxed{h(n) = \frac{3}{4} \left(-\frac{1}{2}\right)^n + \frac{3}{8} \left(\frac{1}{4}\right)^n} \quad \text{for } n \geq 0.$$

BIBO stability

$$x(n) \rightarrow \boxed{\begin{array}{c} h(n) \\ \text{Causal} \end{array}} \rightarrow y(n) = - \sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m)$$

We know that for BIBO stability we must have

$$\sum_{n=0}^{\infty} |h(n)| < \infty$$

But from the Difference Equation (when $N > M$) we obtain that

$$h(n) = \sum_{k=1}^N c_k \lambda_k^n \quad \text{for } n \geq 0$$

Thus

$$\sum_{n=0}^{\infty} \left| \sum_{k=1}^N c_k \lambda_k^n \right| < \sum_{n=0}^{\infty} \sum_{k=1}^N |c_k| |\lambda_k|^n$$

But

$$\begin{aligned} \sum_{n=0}^{\infty} \sum_{k=1}^N |c_k| |\lambda_k|^n &= \sum_{k=1}^N |c_k| \sum_{n=0}^{\infty} |\lambda_k|^n = \\ &= \sum_{k=1}^N |c_k| \left[\frac{1 - |\lambda_k|^{\infty}}{1 - |\lambda_k|} \right] \stackrel{?}{<} \infty \end{aligned}$$

Thus we must have $|\lambda_k| < 1$ for $1 \leq k \leq N$ to have BIBO stable system.

Block Diagram for LTI, DT Systems

$x(n] \rightarrow \boxed{D} \rightarrow x(n-1) ; D \text{ is unit time Delay.}$

Ex 1

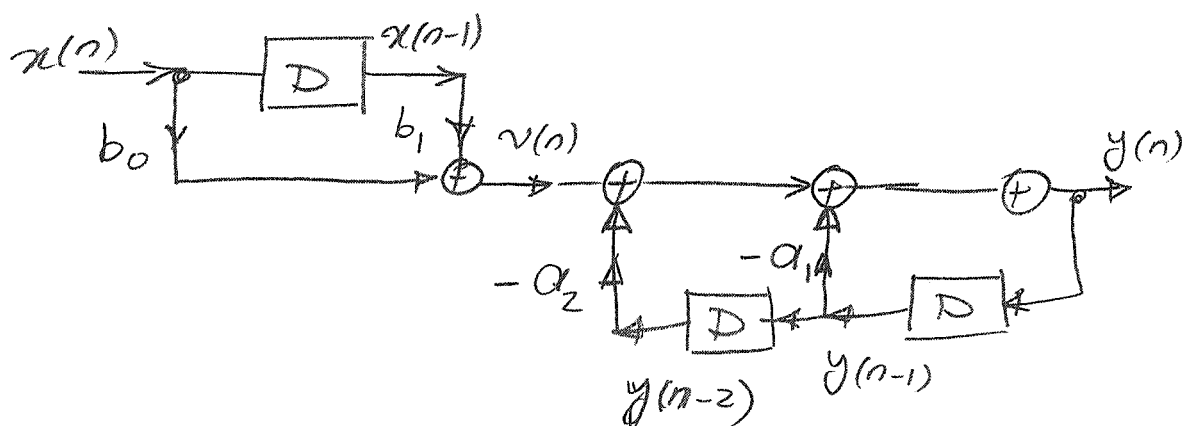
Draw block diagram of the system

$$y(n) = -a_1 y(n-1) - a_2 y(n-2) + b_0 x(n) + b_1 x(n-1).$$

We can write the above difference equation as

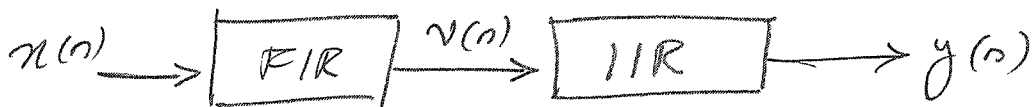
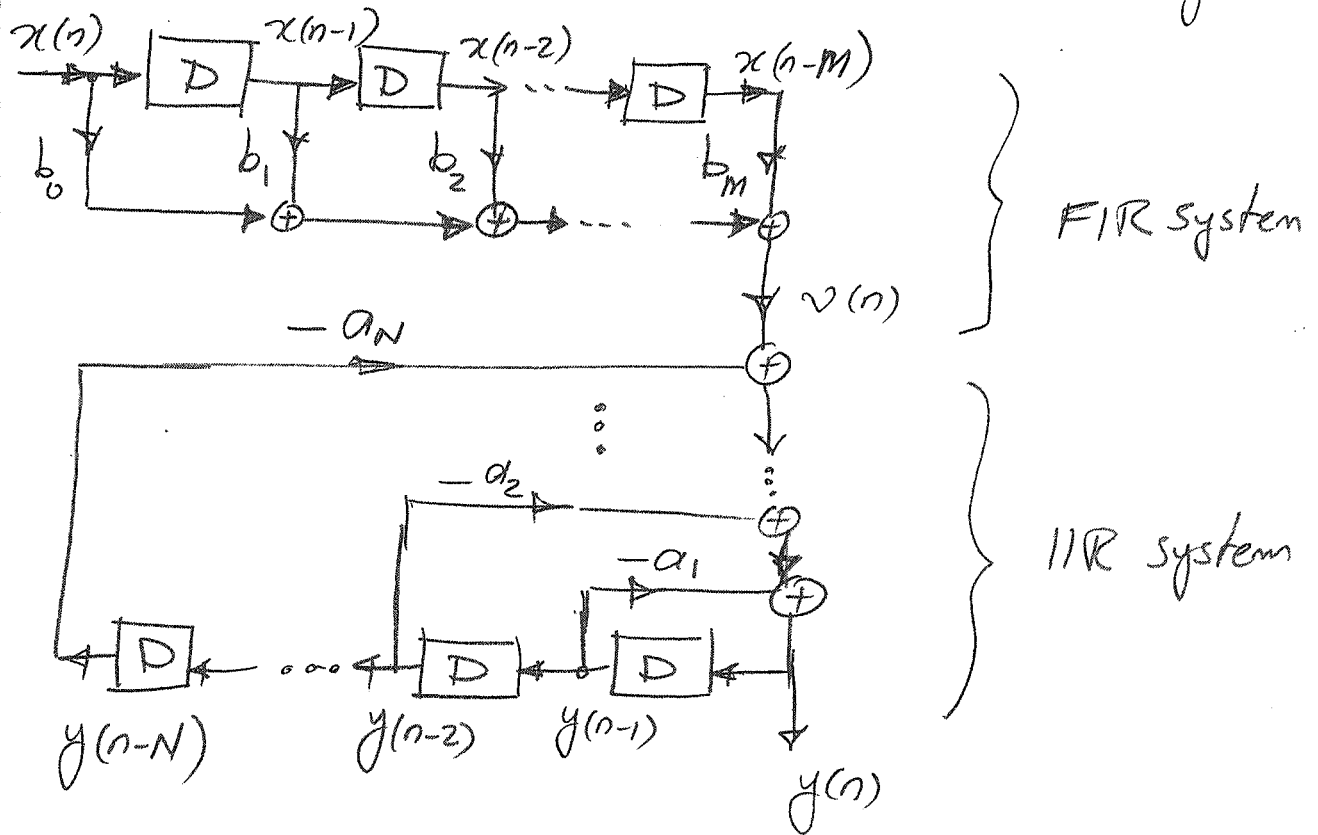
$$\begin{cases} v(n) = b_0 x(n) + b_1 x(n-1) & ; \text{FIR System} \\ y(n) = -a_1 y(n-1) - a_2 y(n-2) + v(n) & ; \text{IIR System} \end{cases}$$

So

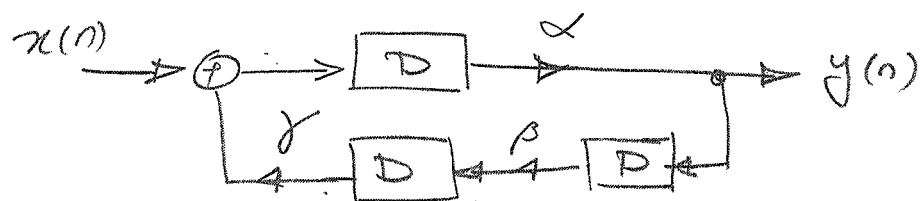


Thus for $y(n) = -\sum_{k=1}^N a_k y(n-k) + \sum_{m=0}^M b_m x(n-m)$
we have

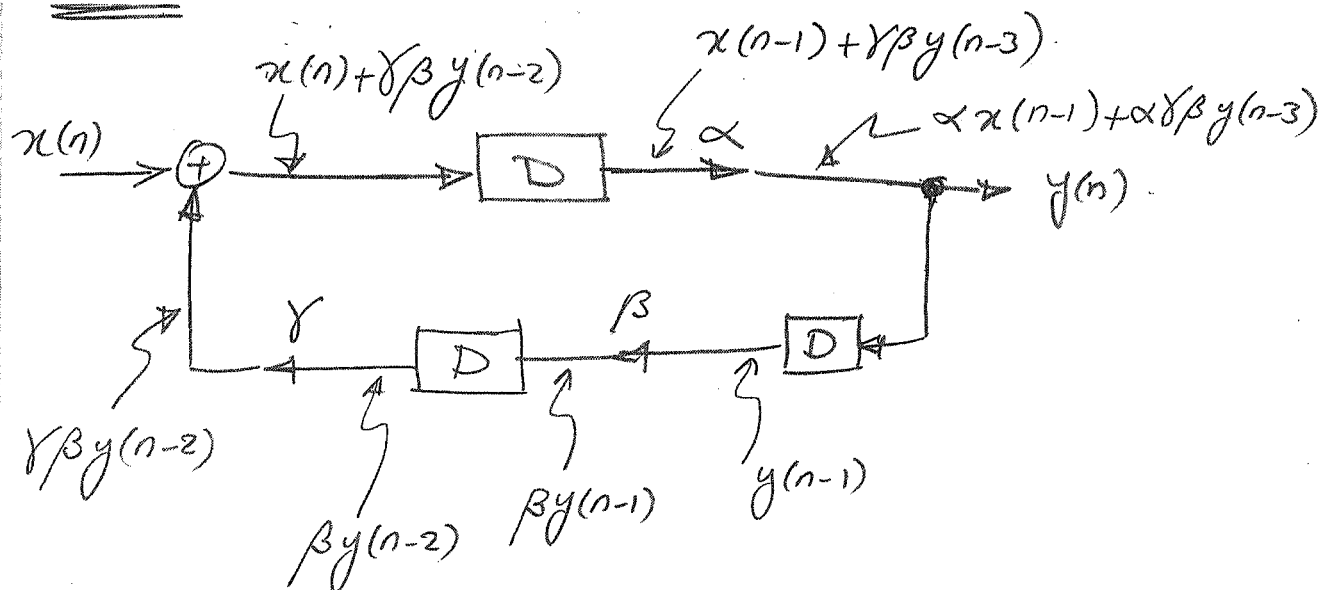
$$\begin{cases} v(n) = \sum_{m=0}^M b_m x(n-m) & \text{: FIR system} \\ y(n) = -\sum_{k=1}^N a_k y(n-k) + v(n) & \text{: IIR system} \end{cases}$$



Ex 2 Find the difference equation for the system shown below.



Sol



Thus
$$y[n] = \alpha \gamma \beta y[n-3] + \alpha x[n-1]$$